Three-Dimensional Figures

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The number of coins created by the U.S. Mint changes each year. In the year 2000, there were about 28 billion coins created—and about half of them were pennies!
Chapter 4 Overview

This chapter focuses on three-dimensional figures. The first two lessons address rotating and stacking two-dimensional figures to create three-dimensional solids. Cavalieri’s principle is presented and is used to derive the formulas for a volume of a cone, pyramid, and sphere. Students then investigate total and lateral surface area of solid figures and use these formulas, along with volume formulas, to solve problems. The chapter culminates with the topics of cross sections and diagonals in three dimensions.

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# Skills Practice Correlation for Chapter 4

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Whirlygigs for Sale!

Rotating Two-Dimensional Figures through Space

**LEARNING GOALS**

In this lesson, you will:

- Apply rotations to two-dimensional plane figures to create three-dimensional solids.
- Describe three-dimensional solids formed by rotations of plane figures through space.

**ESSENTIAL IDEAS**

- Rotations are applied to two-dimensional plane figures.
- Three-dimensional solids are formed by rotations of plane figures through space.

**KEY TERM**

- disc

**TEXAS ESSENTIAL KNOWLEDGE AND SKILLS FOR MATHEMATICS**

(10) Two-dimensional and three-dimensional figures. The student uses the process skills to recognize characteristics and dimensional changes of two- and three-dimensional figures. The student is expected to:

(A) identify the shapes of two-dimensional cross-sections of prisms, pyramids, cylinders, cones, and spheres and identify three-dimensional objects generated by rotations of two-dimensional shapes.
Overview
Models of two-dimensional figures are rotated through space. Students analyze the three-dimensional solid images associated with the rotation. Technically, the rotation of a single point or collection of points changes the location of the point or collection of points. Applied to the rotating pencil activities in this lesson, it is not an actual solid that results from rotating the pencil, rather an image to the eye that is associated with this motion.
Warm Up

1. List objects you have seen that spin but do not require batteries.
   
   Answers will vary.
   
   A top, a gyroscope, a jack, a ball, a coin, a yo-yo

2. Describe how these toys are able to spin.
   
   These objects are powered by energy sources such as flicking a wrist, twirling a few fingers, or pulling a string.
Throughout this chapter, you will analyze three-dimensional objects and solids that are “created” through transformations of two-dimensional plane figures.

But, of course, solids are not really “created” out of two-dimensional objects. How could they be? Two-dimensional objects have no thickness. If you stacked a million of them on top of each other, their combined thickness would still be zero. And translating two-dimensional figures does not really create solids. Translations simply move a geometric object from one location to another.

However, thinking about solid figures and three-dimensional objects as being “created” through transformations of two-dimensional objects is useful when you want to see how volume formulas were “created.”

**Learning Goals**

- Apply rotations to two-dimensional plane figures to create three-dimensional solids.
- Describe three-dimensional solids formed by rotations of plane figures through space.

**Key Term**

- disc
**Problem 1**

A scenario is used which prompts students to tape a rectangle to a pencil and rotate the pencil. They identify the three-dimensional solid image associated with this rotation as a cylinder and relate the dimensions of the rectangle to the dimensions of the image of the solid. This activity is repeated with a circle, and a triangle.

**Grouping**

- Ask students to read the information. Discuss as a class.
- Have students complete Questions 1 with a partner. Then have students share their responses as a class.

**Guiding Questions for Share Phase, Question 1**

- If the rectangle was turned lengthwise and then taped, how would that affect the image of the solid associated with the rotation?
- If the rectangle wasn’t taped in the middle, but taped along a side, how would that affect the image of the solid associated with the rotation?
- If the rectangle was taped in a diagonal fashion to the pencil, how would that affect the image of the solid associated with the rotation?
- Will the image associated with this rotation always be a cylinder?

**PROBLEM  Rectangular Spinners**

You and a classmate are starting a summer business, making spinning toys for small children that do not require batteries and use various geometric shapes. Previously, you learned about rotations on a coordinate plane. You can also perform rotations in three-dimensional space.

1. You and your classmate begin by exploring rectangles.
   a. Draw a rectangle on an index card.
   b. Cut out the rectangle and tape it along the center to a pencil below the eraser as shown.
   c. Hold on to the eraser with your thumb and index finger such that the pencil is resting on its tip. Rotate the rectangle by holding on to the eraser and spinning the pencil. You can get the same effect by putting the lower portion of the pencil between both palms of your hands and rolling the pencil by moving your hands back and forth.
   d. As the rectangle rotates about the pencil, the image of a three-dimensional solid is formed. Which of these solids most closely resembles the image formed by the rotating rectangle?

   ![Figure 1](image1.png)  ![Figure 2](image2.png)  ![Figure 3](image3.png)  ![Figure 4](image4.png)

   The image formed by the rotating rectangle closely resembles Figure 2.

   e. Name the solid formed by rotating the rectangle about the pencil.
   
   The solid formed by rotating the rectangle about the pencil is a cylinder.

   f. Relate the dimensions of the rectangle to the dimensions of this solid.
   
   The width of the rectangle is the radius of the cylinder. The length of the rectangle is the height of the cylinder.
Grouping
Have students complete Question 2 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 2

- If the circle was turned 90 degrees and then taped, how would that affect the image of the solid associated with the rotation?
- If the circle wasn’t taped in the middle, but taped slightly to the right or to the left of the middle, how would that affect the image of the solid associated with the rotation?
- If the circle was taped along its circumference to the pencil, how would that affect the image of the solid associated with the rotation?
- Will the image associated with this rotation always be a sphere?

2. You and your classmate explore circles next.
   
   a. Draw a circle on an index card.
   
   b. Cut out the circle and tape it along the center to a pencil below the eraser as shown.
   
   c. Hold on to the eraser with your thumb and index finger such that the pencil is resting on its tip. Rotate the circle by holding on to the eraser and spinning the pencil. You can get the same effect by putting the lower portion of the pencil between both palms of your hands and rolling the pencil by moving your hands back and forth.

   Remember, a circle is the set of all points that are equal distance from the center. A disc is the set of all points on the circle and in the interior of the circle.

   d. As the disc rotates about the pencil, the image of a three-dimensional solid is formed. Which of these solids most closely resembles the image formed by the rotating disc?

   ![Figure 1](image1.png) ![Figure 2](image2.png) ![Figure 3](image3.png) ![Figure 4](image4.png)

   The image formed by the rotating disc closely resembles Figure 4.

   e. Name the solid formed by rotating the circle about the pencil. The solid formed by rotating the circle about the pencil is a sphere.

   f. Relate the dimensions of the disc to the dimensions of this solid. The radius of the disc is also the radius of the sphere.
Grouping
Have students complete Question 3 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Question 3
- If the triangle was turned sideways and then taped, how would that affect the image of the solid associated with the rotation?
- If the triangle wasn’t taped in the middle, but taped along a side, how would that affect the image of the solid associated with the rotation?
- If the triangle was taped upside down to the pencil, how would that affect the image of the solid associated with the rotation?
- Will the image associated with this rotation always be a cone?

3. You and your classmate finish by exploring triangles.
   a. Draw a triangle on an index card.
   b. Cut out the triangle and tape it lengthwise along the center to a pencil below the eraser as shown.
   c. Hold on to the eraser with your thumb and index finger such that the pencil is resting on its tip. Rotate the triangle by holding on to the eraser and spinning the pencil. You can get the same effect by putting the lower portion of the pencil between both palms of your hands and rolling the pencil by moving your hands back and forth.
   d. As the triangle rotates about the pencil, the image of a three-dimensional solid is formed. Which of these solids most closely resembles the image formed by the rotating triangle?

   ![Figure 1](image1.png)  ![Figure 2](image2.png)  ![Figure 3](image3.png)  ![Figure 4](image4.png)

   The image formed by the rotating triangle closely resembles Figure 3.

   e. Name the solid formed by rotating the triangle about the pencil.
   The solid formed by rotating the triangle about the pencil is a cone.

   f. Relate the dimensions of the triangle to the dimensions of this solid.
   The base of the triangle is the diameter of the cone. The height of the triangle is the height of the cone.

Be prepared to share your solutions and methods.
## Check for Students’ Understanding

Associate a word in the first column to a word in the second column. Explain your reasoning.

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**Triangle—Cone**

The image of a cone can be visualized by rotating a triangle on a pencil.

**Rectangle—Cylinder**

The image of a cylinder can be visualized by rotating a rectangle on a pencil.

**Circle—Sphere**

The image of a sphere can be visualized by rotating a circle on a pencil.
Cakes and Pancakes
Translating and Stacking Two-Dimensional Figures

LEARNING GOALS

In this lesson, you will:
- Apply translations to two-dimensional plane figures to create three-dimensional solids.
- Describe three-dimensional solids formed by translations of plane figures through space.
- Build three-dimensional solids by stacking congruent or similar two-dimensional plane figures.

KEY TERMS

- isometric paper
- right triangular prism
- oblique triangular prism
- right rectangular prism
- oblique rectangular prism
- right cylinder
- oblique cylinder

ESSENTIAL IDEAS

- Rigid motion is used in the process of redrawing two-dimensional plane figures as three-dimensional solids.
- Models of three-dimensional solids are formed using translations of plane figures through space.
- Models of two-dimensional plane figures are stacked to create models of three-dimensional solids.

TEXAS ESSENTIAL KNOWLEDGE AND SKILLS FOR MATHEMATICS

(11) Two-dimensional and three-dimensional figures. The student uses the process skills in the application of formulas to determine measures of two- and three-dimensional figures. The student is expected to:

(C) apply the formulas for the total and lateral surface area of three-dimensional figures, including prisms, pyramids, cones, cylinders, spheres, and composite figures, to solve problems using appropriate units of measure

(D) apply the formulas for the volume of three-dimensional figures, including prisms, pyramids, cones, cylinders, spheres, and composite figures, to solve problems using appropriate units of measure
Overview
Models of two-dimensional figures are translated in space using isometric dot paper. Students analyze the three-dimensional solid images associated with the translations. Technically, the translation of a single point or collection of points changes the location of the point or collection of points. Applied to the activities in this lesson, it is not an actual solid that results from translating the plane figure, rather an image to the eye that is associated with this movement. The activity that includes stacking the two-dimensional models should provide opportunities for these types of discussions as well. Both right and oblique prisms and cylinders are used in this lesson.
Warm Up

1. Translate the triangle on the coordinate plane in a horizontal direction.

2. Describe the translation of each vertex.
   Answers will vary.
   Each vertex was moved to the right 5 units.

3. Translate the triangle on the coordinate plane in a vertical direction.
4. Describe the translation of each vertex.
   Answers will vary.
   Each vertex was moved to the up 3 units.

5. Translate the triangle on the coordinate plane in a diagonal direction.

6. Describe the translation of each vertex.
   Answers will vary.
   Each vertex was moved to the right 5 units and up 3 units.

7. What do you suppose is the difference between translating a triangle on a coordinate plane, as you have done in the previous questions, and translating a triangle through space?
   Answers will vary.
   When a figure is translated on a coordinate plane, the vertices are moved in two directions, left or right, and up or down. When a figure is translated through space, space is three-dimensional so the vertices move in three directions, left or right, up or down, and backward or forward.
LEARNING GOALS

In this lesson, you will:

- Apply translations to two-dimensional plane figures to create three-dimensional solids.
- Describe three-dimensional solids formed by translations of plane figures through space.
- Build three-dimensional solids by stacking congruent or similar two-dimensional plane figures.

KEY TERMS

- isometric paper
- right triangular prism
- oblique triangular prism
- right rectangular prism
- oblique rectangular prism
- right cylinder
- oblique cylinder

You may never have heard of isometric projection before, but you have probably seen something like it many times when playing video games.

Isometric projection is used to give the environment in a video game a three-dimensional effect by rotating the visuals and by drawing items on the screen using angles of perspective.

One of the first uses of isometric graphics was in the video game Q*bert, released in 1982. The game involved an isometric pyramid of cubes. The main character, Q*bert, starts the game at the top of the pyramid and moves diagonally from cube to cube, causing them to change color. Each level is cleared when all of the cubes change color. Of course, Q*bert is chased by several enemies.

While it may seem simple now, it was extremely popular at the time. Q*bert had his own line of toys, and even his own animated television show!
**Problem 1**
A scenario in which students use isometric paper to create the images of three-dimensional solids on two-dimensional paper is the focus of this problem. The three-dimensional solids highlighted in this problem are both right and oblique: triangular prisms, rectangular prisms, and cylinders. Students conclude that the lateral faces of prisms are parallelograms. Rigid motion helps students visualize how models of solids can be formed from models of plane figures.

**Grouping**
Have students complete Questions 1 through 4 with a partner. Then have students share their responses as a class.

**Guiding Questions for Share Phase, Questions 1 through 4**
- Are the sides or lateral faces formed by parallel lines? How do you know?
- What is a prism?
- How are prisms named?
- What is a triangular prism?
- What are some properties of a triangular prism?
- How many units and in what direction did you translate the triangle on the isometric paper?
- When you connected the corresponding vertices, do the sides appear to be parallel to each other?
- Do the lateral faces appear to be parallelograms? How do you know?
- Are you translating the triangle through space in a direction that is perpendicular to the plane containing the triangle? How do you know?
- What is the difference between an oblique triangular prism and a right triangular prism?

**PROBLEM These Figures Take the Cake**
You can translate a two-dimensional figure through space to create a model of a three-dimensional figure.

1. Suppose you and your classmate want to design a cake with triangular bases. You can imagine that the bottom triangular base is translated straight up to create the top triangular base.

   ![Diagram of a two-dimensional figure being translated into a three-dimensional figure.](image)

   a. What is the shape of each lateral face of this polyhedron formed by this translation?
   Each lateral face is a rectangle.

   b. What is the name of the solid formed by this translation?
   The solid formed by this translation is a triangular prism.

A two-dimensional representation of a triangular prism can be obtained by translating a triangle in two dimensions and connecting corresponding vertices. You can use isometric paper, or dot paper, to create a two-dimensional representation of a three-dimensional figure. Engineers often use isometric drawings to show three-dimensional diagrams on “two-dimensional” paper.
2. Translate each triangle to create a second triangle. Use dashed line segments to connect the corresponding vertices.
   a. Translate this triangle in a diagonal direction.

   b. Translate this right triangle in a diagonal direction.

   c. Translate this triangle vertically.

   d. Translate this triangle horizontally.

3. What do you notice about the relationship among the line segments connecting the vertices in each of your drawings?
   The line segments appear to be parallel to each other, and they are congruent.
Grouping

Have students complete Questions 5 through 9 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 5 through 9

- Are the sides or lateral faces formed by parallel lines? How do you know?
- What is the difference between a triangular prism and a rectangular prism?
- What do a triangular prism and rectangular prism have in common?
- What are some properties of a rectangular prism?
- How is isometric dot paper different than Cartesian graph paper?
- How does isometric dot paper help visualize three dimensions?
- How many units and in what direction did you translate the rectangle on the isometric paper?
- When you connected the corresponding vertices, do the sides appear to be parallel to each other?
- Do the lateral faces appear to be parallelograms? How do you know?
- Are you translating the rectangle through space in a direction that is perpendicular to the plane containing the rectangle? How do you know?
- What is the difference between an oblique rectangular prism and a right rectangular prism?

When you translate a triangle through space in a direction that is perpendicular to the plane containing the triangle, the solid formed is a right triangular prism. The triangular prism cake that you and your classmate created in Question 1 is an example of a right triangular prism. When you translate a triangle through space in a direction that is not perpendicular to the plane containing the triangle, the solid formed is an oblique triangular prism. An example of an oblique triangular prism is shown.

4. What is the shape of each lateral face of an oblique triangular prism?
   Each lateral face is a parallelogram.

5. Suppose you and your classmate want to design a cake with rectangular bases. You can imagine that the bottom rectangular base is translated straight up to create the top rectangular base.
   a. What is the shape of each lateral face of the solid figure formed by this translation?
      Each lateral face would be a rectangle.
   b. What is the name of the solid formed by this translation?
      The solid formed by this translation is a rectangular prism.
A two-dimensional representation of a rectangular prism can be obtained by translating a rectangle in two dimensions and connecting corresponding vertices.

6. Draw a rectangle and translate it in a diagonal direction to create a second rectangle. Use dashed line segments to connect the corresponding vertices.

7. Analyze your drawing.
   a. What do you notice about the relationship among the line segments connecting the vertices in the drawing?
      The line segments appear to be parallel to each other, and they are congruent.
   b. What is the name of a rectangular prism that has all congruent sides?
      A rectangular prism with all congruent sides is a cube.
   c. What two-dimensional figure would you translate to create a rectangular prism with all congruent sides?
      To create a rectangular prism with all congruent sides, I would translate a square.
   d. Sketch an example of a rectangular prism with all congruent sides.
When you translate a rectangle through space in a direction that is perpendicular to the plane containing the rectangle, the solid formed is a right rectangular prism. The rectangular prism cake that you and your classmate created in Question 8 is an example of a right rectangular prism. When you translate a rectangle through space in a direction that is not perpendicular to the plane containing the rectangle, the solid formed is an oblique rectangular prism.

8. What shape would each lateral face of an oblique rectangular prism be?
   Each lateral face would be a parallelogram.

9. Sketch an oblique rectangular prism.

10. Suppose you and your classmate want to design a cake with circular bases. You can imagine that the bottom circular base, a disc, is translated straight up to create the top circular base.

   a. What shape would the lateral face of the solid figure formed by this translation be?
      The lateral face would be a rectangle.

   b. What is the name of the solid formed by this translation?
      The solid formed by this translation is a cylinder.

Grouping
Have students complete Questions 10 through 14 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 10 through 14

- Are the sides or lateral faces formed by parallel lines? How do you know?
- What is the difference between a prism and a cylinder?
- What do a prism and cylinder have in common?
- What are some properties of a cylinder?
- When you connected the corresponding tops and bottoms of the cylinder, do the sides appear to be parallel to each other?

- Do the lateral faces appear to be parallelograms? How do you know?
- Are you translating a circle through space in a direction that is perpendicular to the plane containing the circle? How do you know?
- What is the difference between an oblique cylinder and a right cylinder?
A two-dimensional representation of a cylinder can be obtained by translating an oval in two dimensions and connecting the tops and bottoms of the ovals.

11. Translate the oval in a diagonal direction to create a second oval. Use dashed line segments to connect the tops and bottoms of the ovals.

12. What do you notice about the relationship among the line segments in the drawing? The line segments appear to be parallel to each other, and they are congruent.

When you translate a disc through space in a direction that is perpendicular to the plane containing the disc, the solid formed is a right cylinder. The cylinder cake that you and your classmate created in Question 13 is an example of a right cylinder. When you translate a disc through space in a direction that is not perpendicular to the plane containing the disc, the solid formed is an oblique cylinder.

13. What shape would the lateral face of an oblique cylinder be? The lateral face would be a parallelogram.

Problem 2
Within the context of a situation, students stack a variety of shapes to determine the solid formed. At first they stack congruent circles, congruent triangles, and congruent rectangles. Then they stack circles of different sizes, squares of different sizes, and other similar polygons. The solids determined in this problem include cylinders, prisms, cones, and pyramids.

Grouping
Have students complete Questions 1 through 5 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 5
- What information do you need to know about one pancake to determine the height of the stack of pancakes?
- What unit of measure is easiest to use when measuring the height of a pancake?
- What do you suppose is the approximate height of one pancake?
- How many pancakes would be a reasonable portion? Why?
- Is the radius of the disc the same or different than the radius of the cylinder?
- Is the height of the stack of discs the same or different than the height of the cylinder?
- Is the length of a side of the square pancake a consideration when determining the height of the stack of pancakes? Why or why not?
- Do you suppose these square pancakes are larger or smaller than the circular pancakes? Why?
- How many pancakes would be a reasonable portion? Why?

PROBLEM 2 Congruent and Similar
The math club at school is planning a pancake breakfast as a fund-raiser. Because this is a fund-raiser for the math club, the pancakes will use various geometric shapes!

1. Imagine you stack congruent circular pancakes on top of each other.
   - What is the name of the solid formed by this stack of pancakes?
     The solid formed by stacking congruent circular pancakes is a cylinder.
   - Relate the dimensions of a single circular pancake to the dimensions of this solid.
     The radius of the circular pancakes is the radius of the cylinder. The height of the stack of circular pancakes is the height of the cylinder.

2. Imagine you stack congruent square pancakes on top of each other.
   - What is the name of the solid formed by this stack of pancakes?
     The solid formed by stacking congruent square pancakes is a right rectangular prism.
   - Relate the dimensions of a single square to the dimensions of this solid.
     The length of the side of a square pancake is also the length and width of the right rectangular prism. The height of the stack of pancakes is the height of the right rectangular prism.

Remember, similar figures have the same shape. Congruent figures have the same shape AND size.
3. Imagine you stack congruent triangular pancakes on top of each other.

a. What is the name of the solid formed by this stack of pancakes?
   The solid formed by stacking congruent triangular pancakes is a triangular prism.

b. Relate the dimensions of the triangle to the dimensions of this solid.
   The dimensions of the triangular pancake (length of the base and the height) are also the dimensions of the base of the triangular prism. The height of the stack of pancakes is the height of the triangular prism.

4. What do you notice about the three-dimensional solids created by stacking congruent figures?
   Stacking congruent circles creates a right cylinder.
   Stacking congruent triangles or squares creates a right prism.

5. What type of solid would be formed by stacking congruent rectangles? pentagons? hexagons?
   Stacking congruent polygons creates a right prism. The shape of the figure being stacked is the shape of the base of the prism.
Grouping
Have students complete Questions 6 through 12 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 6 through 12
- How is the stack of similar square pancakes different than the stack of congruent square pancakes?
- How is the stack of similar equilateral triangular pancakes different than the stack of congruent triangular pancakes?
- What is the name of a pyramid with a pentagonal base?
- How is the base of the pyramid related to the dimensions of the pyramid?
- What is the name of a pyramid with a hexagonal base?
- How many hexagons do you suppose are needed to create a hexagonal pyramid?
- What do you suppose would be the radius of the topmost disc?
- How is this stack of similar discs different than the stack of congruent discs?

6. Imagine you stack similar circular pancakes on top of each other so that each layer of the stack is composed of a slightly smaller pancake than the previous layer.
   a. What is the name of the solid formed by this stack of pancakes?
      The solid formed by stacking similar circular pancakes is a cone.
   b. Relate the dimensions of a single pancake to the dimensions of the solid.
      The radius of the pancake at the bottom layer is the radius of the cone. The height of the stack of pancakes is the height of the cone.

7. Imagine you stack similar square pancakes on top of each other so that each layer of the stack is composed of a slightly smaller pancake than the previous layer.
   a. What is the name of the solid formed by this stack of pancakes?
      The solid formed by stacking similar square pancakes is a rectangular pyramid.
   b. Relate the dimensions of a single pancake to the dimensions of the solid.
      The length and width of the pancake at the bottom layer is the length and width of the base of the pyramid. The height of the stack of pancakes is the height of the pyramid.

8. Imagine you stack similar triangular pancakes on top of each other so that each layer of the stack is composed of a slightly smaller pancake than the previous layer.
   a. What is the name of the solid formed by this stack of pancakes?
      The solid formed by stacking similar triangular pancakes is a triangular pyramid.
   b. Relate the dimensions of a single pancake to the dimensions of the solid.
      The dimensions of the pancake at the bottom layer are the dimensions of the base of the pyramid. The height of the stack of pancakes is the height of the pyramid.
9. What do you notice about the three-dimensional solids created by stacking similar figures?
Stacking similar circles creates a cone.
Stacking similar triangles or squares creates a pyramid.

10. What type of solid would be formed by stacking similar rectangles? pentagons? hexagons?
Stacking congruent polygons creates a pyramid. The shape of the figure being stacked is the shape of the base of the pyramid.

11. Use what you have learned in this lesson to make an informal argument that explains the volume formulas for prisms and cylinders.
The base of a cylinder is a circle with an area of $\pi r^2$. When this area is translated a distance of $h$, the resulting formula is $\pi r^2 h$.
The base of a prism is a polygon. The area of the base depends on the type of polygon, but I can denote this area with $B$. The prism is created by translating (or stacking) the area of the base a distance (or height) of $h$. So, the resulting volume formula is $Bh$.

12. Complete the graphic organizer to record the volume formulas and the transformations you have used to create the solid figures.
As students learn and derive the volume formulas, review the terms *base* and *height*. Discuss how to calculate the area of common two-dimensional geometric figures such as rectangles, triangles, and circles.
Grouping
Have students complete Questions 13 and 14 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 13 and 14
• What is the shape of the bases of the container of milk?
• What is the shape of the lateral faces of the container?
• What is the height of the chocolate package?

13. A candy company is deciding whether to begin selling chocolate milk at its store in containers that are triangular prisms. The milk could pour from a hole in one corner of the top. What is the volume of the container of milk shown at the right?
   a. Determine the area of each congruent base.
      The area of each base is \( \frac{1}{2} \times 3 \times 4 \), or 6 square inches
   b. Apply the formula \( V = Bh \) to determine the volume of the triangular prism.
      The volume of the triangular prism is 60 in.\(^3\)
      \[ V = (6)(10) = 60 \]

14. A certain kind of imported chocolate comes in packages that are shaped like triangular prisms, as shown. What volume of chocolate does the container hold?
   The package holds about 320 cubic centimeters of chocolate.
   \[ V = \left( \frac{1}{2} \right) \times 8 \times 4 \times 20 = 320 \]
Talk the Talk

Students are given several different actions. Some of the actions could result in forming the same solid. They cut out separate cards containing each action and sort them into groups such that each group forms the same solid. They also label each group using the name of the solid and draw an example.

Grouping

Have students complete Question 1 with a partner. Then have students share their responses as a class.

ELL Tip

Have students work in pairs as they investigate each card. Students should take turns explaining to each other how they decided on the resulting solid. Encourage students to self-correct as they speak to each other by responding to cues from their partner.

**Talk the Talk**

1. Which of the following actions could result in forming the same solid? Cut out the cards shown and sort them into groups that could each form the same solid figure. Then, draw an example of each solid figure and label each group. Explain how you sorted the actions. Be sure to name the solid that best represents the object.

- translating an isosceles triangle
- translating a right triangle
- translating a square
- translating a rectangle
- translating a circle
- rotating a rectangle
- rotating a triangle
- rotating a circle
- stacking congruent circles
- stacking similar circles
- stacking congruent rectangles
- stacking similar rectangles
- stacking congruent squares
- stacking similar squares
- stacking congruent triangles
- stacking similar triangles
Group 1: Sphere
- rotating a circle
- stacking similar circles

Group 2: Cone
- rotating a triangle
- stacking similar circles

Group 3: Cylinder
- rotating a rectangle
- translating a circle
- stacking congruent circles

Group 4: Pyramid
- stacking similar triangles
- stacking similar squares
- stacking similar rectangles

Group 5: Prism
- translating a square
- stacking congruent squares
- translating a rectangle
- stacking congruent rectangles
- translating an isosceles triangle
- translating a right triangle
- stacking congruent triangles

Be prepared to share your solutions and methods.
Check for Students’ Understanding

1. Name the geometric solid associated with each situation.
   a. A stack of 100 copies of the same book
      rectangular prism
   b. A stack of 100 tires. The tires range in size from very large tires to very small tires
      cone
   c. A stack of 100 stop signs
      octagonal prism

2. Create a stacking situation that best resembles each geometric solid.
   a. A cylinder
      Answers will vary.
      Example Response: one hundred music CDs
   b. A square pyramid
      Answers will vary.
      Example Response: fifty similar progressively smaller squares
   c. A triangular prism
      Answers will vary.
      Example Response: forty yield signs
ESSENTIAL IDEAS

- If two regions in a plane are located between two parallel lines in the plane and all the lines parallel to these two lines intersects both regions in line segments of equal length, then the area of the two regions are equal.

- If two solids of equal altitude, and the sections formed by planes parallel to and at the same distance from their bases are equal, then the volumes of the two solids are equal.

KEY TERM

- Cavalieri’s principle

TEXAS ESSENTIAL KNOWLEDGE AND SKILLS FOR MATHEMATICS

(11) Two-dimensional and three-dimensional figures. The student uses the process skills in the application of formulas to determine measures of two- and three-dimensional figures. The student is expected to:

(D) apply the formulas for the volume of three-dimensional figures, including prisms, pyramids, cones, cylinders, spheres, and composite figures, to solve problems using appropriate units of measure.
Overview

Students approximate the area of an irregularly shaped figure by dividing the figure into familiar polygons and finish with showing Cavalieri’s principle for two-dimensional figures; if the lengths of one-dimensional slices—just a line segment—of the two figures are the same, then the figures have the same area. Next, students first approximate the volume of a right rectangular prism and an oblique rectangular prism and end with showing Cavalieri’s principle for three-dimensional figures; if, in two solids of equal altitude, the sections made by planes parallel to and at the same distance from their respective bases are always equal, then the volumes of the two solids are equal.
Warm Up

Consider rectangle $ABCD$ and parallelogram $EFGH$.

1. What is the length of every line segment drawn in the interior of rectangle $ABCD$ and parallel to line segments $AB$ and $CD$?
   The every line segment drawn in the interior of rectangle $ABCD$ and parallel to line segments $AB$ and $CD$ is equal to 13 cm.

2. What is the length of every line segment drawn in the interior of parallelogram $EFGH$ and parallel to line segments $EF$ and $GH$?
   The every line segment drawn in the interior of parallelogram $EFGH$ and parallel to line segments $EF$ and $GH$ is equal to 13 cm.

3. Compare the area of rectangle $ABCD$ to the area of parallelogram $EFGH$.
   The area of rectangle $ABCD$ and the area of parallelogram $EFGH$ are both equal to 78 square centimeters.
Bonaventura Cavalieri was an Italian mathematician who lived from 1598 to 1647. Cavalieri is well known for his work in geometry as well as optics and motion.

His first book dealt with the theory of mirrors shaped into parabolas, hyperbolas, and ellipses. What is most amazing about this work is that the technology to create the mirrors that he was writing about didn’t even exist yet!

Cavalieri is perhaps best known for his work with areas and volumes. He is so well known that he even has a principle named after him—Cavalieri’s principle.

LEARNING GOALS

In this lesson, you will:

- Explore Cavalieri’s principle for two-dimensional geometric figures (area).
- Explore Cavalieri’s principle for three-dimensional objects (volume).

KEY TERM

- Cavalieri’s principle
Problem 1
A strategy for approximating the area of an irregularly shaped figure is given. Students answer questions related to the situation which lead to Cavalieri’s Principle for two-dimensional figures.

Grouping
- Ask a student to read the information aloud. Discuss as a class.
- Have students complete Questions 1 through 5 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 5
- What information do you know about the rectangles dividing the curved figure?
- What is the height of each rectangle if there are 10 rectangles and the total height is represented by \( h \)?
- What do the variables represent in the diagram?
- How is the area of one rectangle determined?
- How is the area of ten rectangles determined?
- If the ten rectangles were arranged differently, would the total area remain the same or change?
- Are the areas of the two figures the same or is one figure larger than the other?

PROBLEM Approximating the Area of a Two-Dimensional Figure

One strategy for approximating the area of an irregularly shaped figure is to divide the figure into familiar shapes and determine the total area of all the shapes. Consider the irregular shape shown. The distance across any part of the figure is the same.

1. You can approximate the area by dividing the irregular shape into congruent rectangles. To start, let’s divide this shape into 10 congruent rectangles.

   a. What is the length, the height, and the area of each congruent rectangle?
   The length of each congruent rectangle is \( \ell \), and the height of each congruent rectangle is \( \frac{h}{10} \). The area of each congruent rectangle is \( \frac{\ell h}{10} \).

   b. What is the approximate area of the irregularly shaped figure?
   The approximate area of the irregularly shaped figure is \( \frac{\ell h}{10} \cdot 10 = \ell h \).

2. If this irregularly shaped figure were divided into 1000 congruent rectangles, what would be the area of each congruent rectangle? What would be the approximate area of the figure?
   The area of each congruent rectangle is \( \frac{\ell h}{1000} \).
   The approximate area of the irregularly shaped figure would be \( \frac{\ell h}{1000} \cdot 1000 = \ell h \).

- Do the two figures contain the same number of rectangles?
- Are all of the rectangles in both figures the same size?
- Can you think of a different example of using Cavalieri’s principle?
3. If this irregularly shaped figure were divided into \( n \) congruent rectangles, what would be the area of each congruent rectangle? What would be the approximate area of the figure?
   The area of each congruent rectangle is \( \frac{\ell h}{n} \).
   The approximate area of the irregularly shaped figure would be \( \frac{\ell h}{n} \cdot n = \ell h \).

4. If the irregularly shaped figure were divided into only one rectangle, what would be the approximate area of the figure?
   The approximate area of the irregularly shaped figure would be \( \frac{\ell h}{1} = \ell h \).

5. Compare the area of the two figures shown. Each rectangle has a height of \( h \) and a base equal to length \( \ell \).
   The approximate area of both figures is \( \frac{\ell h}{6} \cdot 6 = \ell h \).

You have just explored **Cavalieri’s principle** for two-dimensional figures, sometimes called the method of indivisibles. If the lengths of one-dimensional slices—just a line segment—of the two figures are the same, then the figures have the same area. This is best illustrated by making several slices to one figure and pushing them to the side to form a second figure.
Problem 2
A right rectangular prism and an oblique rectangular prism are used to show Cavalieri’s Principle for three-dimensional figures.

Grouping
- Instruct students to read the information. Discuss as a class.
- Have students complete Questions 1 through 4 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 4
- Is the cross section of the right rectangular prism different than the cross section of the oblique rectangular prism?
- Do perpendicular lines intersect at corners of the oblique rectangular prism?
- What do the variables represent in the diagram?
- How is the volume of one slice determined?
- How is the volume of ten slices determined?
- Are the volumes of the two figures the same or is one figure larger than the other?
- Do the two figures contain the same number of slices?

PROBLEM Cavalieri’s Principle for Volume
Consider the right rectangular prism and the oblique rectangular prism shown.

1. What geometric figure represents a cross section of each that is perpendicular to the base?
   A rectangle best represents the cross sections of the prisms.

2. What are the dimensions of one cross section?
   Each rectangular cross section has dimensions \( l \times w \).

3. What is the volume of the right rectangular prism?
   The volume of the right rectangular prism is \( V = l \times w \times h \).

4. Approximate the volume of the oblique rectangular prism by dividing the prism into ten congruent right prisms as shown.

   The volume of one rectangular prism is \( V = l \times w \times \frac{h}{10} \).
   The volume of the oblique rectangular prism is \( V = (l \times w \times \frac{h}{10})10 = l \times w \times h \).

- Are all of the slices the same size?
- Can you think of a different example of using Cavalieri’s principle?
Grouping

Have students complete Questions 5 through 10 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 5 through 10

- Is the cross section of the right cylinder different than the cross section of the cylinder?
- What do the variables represent in the diagram?
- How is the volume of one slice determined?
- How is the volume of ten slices determined?
- Are the areas of the two figures the same or is one figure larger than the other?
- Do the two figures contain the same number of slices?
- Are all of the slices the same size?
- How is the example using rectangular prisms similar to the example using cylinders?
- Can you think of a different example of using Cavalieri’s principle?

You have just explored Cavalieri’s principle for three-dimensional figures. Given two solids included between parallel planes, if every plane cross section parallel to the given planes has the same area in both solids, then the volumes of the solids are equal. In other words, if, in two solids of equal altitude, the sections made by planes parallel to and at the same distance from their respective bases are always equal, then the volumes of the two solids are equal.

For a second example of this principle, consider a right cylinder and an oblique cylinder having the same height and radii of equal length.

5. What geometric figure best represents the ten cross sections of the cylinders?
   A smaller cylinder best represents the cross sections of the cylinders.

6. What are the dimensions of one cross section?
   The dimensions of the smaller cylinder are the radius \( r \), and the height \( \frac{h}{10} \).

7. What is the volume of one cross section?
   The volume of one cross section is \( V = \pi \left( \frac{h}{10} \right) \).

8. What is the volume of the oblique cylinder?
   The volume of the oblique cylinder is \( V = \pi \left( \frac{h}{10} \right) \). \( \cdot 10 = \pi rh \).

9. What is the volume of the right cylinder?
   The volume of the right cylinder is \( V = \pi rh \).

You have just shown the volume of a right cylinder and the volume of an oblique cylinder are equal, provided both cylinders have the same height and radii of equal length.
10. Using Cavalieri’s principle, what can you conclude about the volume of these two cones, assuming the heights are equal and the radii in each cone are congruent?

Using Cavalieri’s principle, I can conclude the volumes of both cones are equal.

It is important to mention that the Cavalieri principles do not compute exact volumes or areas. These principles only show that volumes or areas are equal without computing the actual values. Cavalieri used this method to relate the area or volume of one unknown object to one or more objects for which the area or volume could be determined.

Talk the Talk

Students explain how Cavalieri’s principle is used to determine the area formula for a parallelogram using the area formula of a rectangle.

Grouping

Have students complete Question 1 with a partner. Then have students share their responses as a class.

1. Consider the rectangle and the parallelogram shown to be of equal height with bases of the same length.

Knowing the area formula for a rectangle, how is Cavalieri’s principle used to determine the area formula for the parallelogram?

The area of the rectangle is \( A = bh \). Because the height of both figures is any fixed number, the widths of the parallel segments in the parallelogram have the same length as the corresponding segments in the rectangle. Therefore, the area of the parallelogram must be the same as the area of the rectangle, or \( A = bh \).

Be prepared to share your solutions and methods.
Check for Students’ Understanding

Use the appropriate words from the list to complete each sentence.

area \hspace{1cm} \text{two-dimensional}
congruent \hspace{1cm} \text{three-dimensional}
similar \hspace{1cm} \text{volume}

1. Based on Cavalieri’s principle, the area of a \underline{two-dimensional} figure can be estimated by adding the area of congruent slices of the figure.

2. Based on Cavalieri’s principle, the volume of an oblique cylinder can be estimated by adding the \underline{volume} of congruent discs of the solid.
Chapter 4  Three-Dimensional Figures
Spin to Win
Volume of Cones and Pyramids

LEARNING GOALS

In this lesson, you will:
- Rotate two-dimensional plane figures to generate three-dimensional figures.
- Give an informal argument for the volume of cones and pyramids.

ESSENTIAL IDEAS
- Rotations are applied to two-dimensional plane figures.
- Three-dimensional solids are formed by rotations of plane figures through space.
- An informal argument links Cavalieri’s principle to the formula for the volume of a cylinder.
- An informal argument links Cavalieri’s principle to the formula for the volume of a pyramid.

TEXAS ESSENTIAL KNOWLEDGE AND SKILLS FOR MATHEMATICS

(11) Two-dimensional and three-dimensional figures. The student uses the process skills in the application of formulas to determine measures of two- and three-dimensional figures. The student is expected to:

(C) apply the formulas for the total and lateral surface area of three-dimensional figures, including prisms, pyramids, cones, cylinders, spheres, and composite figures, to solve problems using appropriate units of measure

(D) apply the formulas for the volume of three-dimensional figures, including prisms, pyramids, cones, cylinders, spheres, and composite figures, to solve problems using appropriate units of measure
Overview
Models of two-dimensional figures such as rectangles and triangles are rotated through space on an axis. Students analyze the three-dimensional solid images associated with the rotation. Similar to the activities in the first lesson of this chapter, the rotation of a single point or collection of points changes the location of the point or collection of points. The rotation on the axis does not result in an actual solid, rather an image to the eye that is associated with this rotation. Through rotation or stacking strategies, the volumes of a cylinder, cone and pyramid are explored.
Warm Up

A cylindrical fish tank provides a 360° view!

- The height of the cylindrical fish tank is 30".
- The length of the diameter of the base is 27.5".
- One US gallon is equal to approximately 231 cubic inches.

Calculate the amount of water the tank will hold.

Volume of the cylinder:

\[ V = \pi r^2 h \]

\[ V = (\pi)(13.75)^2 (30) \approx 17809.6875 \text{ in}^3 \]

\[ \frac{17809.6875}{231} \approx 77.1 \text{ gallons} \]
Spin to Win
Volume of Cones and Pyramids

LEARNING GOALS

In this lesson, you will:

- Rotate two-dimensional plane figures to generate three-dimensional figures.
- Give an informal argument for the volume of cones and pyramids.

Imagine that you are, right now, facing a clock and reading the time on that clock—let’s imagine that it’s 2:28.

Now imagine that you are blasted away from that clock at the speed of light, yet you are still able to read the time on it (of course you wouldn’t be able to, really, but this is imagination!).

What time would you see on the clock as you traveled away from it at the speed of light? The light bouncing off the clock travels at the speed of light, so, as you travel farther and farther away from the clock, all you could possibly see was the time on the clock as it was right when you left.

Einstein considered this “thought experiment” among many others to help him arrive at groundbreaking theories in physics.
Problem 1
Rotating a rectangle about an axis and stacking congruent discs are two methods used for creating the image of a cylinder. Students determine the area of an average disc which leads to the volume formula for any cylinder.

Grouping
Ask students to read introduction and worked example. Discuss Questions 1 and 2 as a class. Discuss as a class.

Guiding Questions for Discuss Phase
- How are the formulas for the area of a circle and volume of a cylinder related?
- What is the height of each disc if there are 10 discs and the total height is represented by $h$?
- What is the area of each disc in the cylinder?
- Are all discs in the cylinder congruent?
- What do the variables in the diagram represent?

PROBLEM Building Cylinders

To calculate the volume of the cylinder, I can use the formula $V = \pi r^2 h$.

The volume is $\pi (8^2)(24)$, or approximately 4825.49 cubic centimeters.

To derive the formula for the volume of a cylinder, you can think of the cylinder as an infinite stack of discs, each with an area of $\pi r^2$. These discs are stacked to a height of $h$, the height of the cylinder.

So, the volume of the cylinder is $\pi r^2 \times h$, or $\pi r^2 h$.

2. Can you choose any disc in the cylinder and multiply the area by the height to calculate the volume of the cylinder? Explain your reasoning.

Yes. Every disc that makes up the cylinder is congruent. So, you can choose any disc and multiply its area by the height to calculate the volume of the cylinder.
Another way to think about a cylinder is the rotation of a rectangle about one of its sides as shown. The rotation of the set of points that make up the rectangle forms the cylinder.

To determine the volume of the cylinder, you can multiply the area of the rectangle by the distance that the points of the rectangle rotate. However, the points of the rectangle don’t all rotate the same distance. Consider a top view of the cylinder. The distance that point A rotates is greater than the distance that point B rotates.

You can’t calculate the distance that each point rotates because there are an infinite number of points, each rotating a different distance. But you can use an average, or typical point, of the rectangle.

3. Consider the dot plot shown.

   a. What is the median of the data?
      The median of the data is the middle value when the data are ordered from least to greatest or greatest to least. The median of these data is 5.

   b. Describe how you could determine the median of these data without doing any calculation.
      There are 11 values that are evenly spaced from 0 to 10, so the median should be the exact center value of the data set. There are five values to the left of 5 and five values to the right of 5, so 5 is the median of these data.
Grouping

Have students complete Questions 4 through 8 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 4 through 8

- How is the rotating rectangle related to the volume of the cylinder?
- Do you prefer the infinite stack of discs or the rotating for deriving the volume formula for a cylinder? Explain.

4. What is the location of the average, or typical, point of the rectangle in terms of the radius and the height? Explain your reasoning.

   The average, or typical, point is the center point of the rectangle. The center of the width of the rectangle is \( \frac{1}{2}r \). The center of the height of the rectangle is \( \frac{1}{2}h \). So, the average, or typical, point of the rectangle is located at \( \left( \frac{1}{2}r, \frac{1}{2}h \right) \).

5. What is the area of the rectangle that is rotated?

   The area of the rectangle is \( rh \).

6. Use the average point of the rectangle to calculate the average distance that the points of the rectangle rotate. Explain your reasoning.

   Because the average point of the rectangle is located at \( \frac{1}{2}r \), the average distance that all of the points of the rectangle are rotated is the circumference of a circle with a radius of \( \frac{1}{2}r \).

   \[
   \text{Circumference} = 2\pi \left( \frac{1}{2}r \right) = \pi r
   \]

   The average distance that all of the points of the rectangle are rotated is \( \pi r \).

7. To determine the volume of the cylinder, multiply the area of the rectangle by the average distance that the points of the rectangle rotate. Calculate the volume of the cylinder.

   To determine the volume formula, I multiply the area of the rectangle, \( rh \), by the average distance that all of the points of the rectangle are rotated, \( \pi r \).

   \[
   \pi r (rh) = \pi rh
   \]

8. Compare the volume that you calculated in Question 2 to the volume that you calculated in Question 7. What do you notice?

   Both methods give the formula for the volume of a cylinder, \( \pi rh \).
**Problem 2**
Rotating a right triangle about an axis and stacking an infinite number of similar discs are two methods for creating the image of a cone. The area of an average disc is calculated in which the length of the radius is determined algebraically. This leads to the volume formula for any cone.

**Grouping**
- Ask students to read the introduction and worked examples. Then discuss as a class.
- Have students complete Questions 1 through 5 with a partner. Then have students share their responses as a class.
Let's review how to calculate the coordinates of the centroid of triangle \(ABC\).

Triangle \(ABC\) has vertices at \(A(0, 0)\), \(B(0, 18)\), and \(C(12, 0)\).

First, determine the locations of the midpoints of side \(AB\) and side \(AC\).

- Midpoint of side \(AC\): \(\left(\frac{0 + 12}{2}, \frac{0 + 0}{2}\right)\), or \((6, 0)\)
- Midpoint of side \(AB\): \(\left(\frac{0 + 0}{2}, \frac{0 + 18}{2}\right)\), or \((0, 9)\)

Next, determine an equation representing each median.

- The slope of \(BE\) is \(-3\), and the \(y\)-intercept is 18.
  
  Slope of \(BE\) = \(\frac{0 - 18}{6 - 0}\) = \(-\frac{18}{6}\) = \(-3\)
  
  So, the equation of \(BE\) is \(y = -3x + 18\).

- The slope of \(CD\) is \(-\frac{3}{4}\), and the \(y\)-intercept is 9.
  
  Slope of \(CD\) = \(\frac{0 - 9}{12 - 0}\) = \(-\frac{9}{12}\) = \(-\frac{3}{4}\)
  
  So, the equation of \(CD\) is \(y = -\frac{3}{4}x + 9\).
Solve the system of equations to determine the coordinates of the centroid.

\[
\begin{align*}
3x + 9 &= 3x + 18 \\
\frac{3}{4}x &= -3x + 9 \\
\frac{1}{4}x &= x - 3
\end{align*}
\]

The centroid of the triangle is located at (4, 6).

The centroid of the triangle is located at (4, 6).
Guiding Questions for Share Phase, Questions 1 through 5

1. What is the area of triangle ABC?
   The base of the triangle is 12 units, and the height is 18 units, so the area of the triangle is \( \frac{1}{2} \times 12 \times 18 \), or 108 square units.

2. What is the average distance that all the points of the triangle are rotated? Show your work and explain your reasoning.
   Because the centroid has an \( x \)-coordinate of 4, the average distance that all the points of the triangle are rotated is the circumference of a circle with a radius of 4. So, the average distance all the points are rotated is \( 2\pi(4) \), or \( 8\pi \) units.

3. Determine the volume of the cone. Show your work and explain your reasoning.
   The volume of the cone is the product of the area of the triangle, 108 square units, and the average distance all the points of the triangle are rotated, \( 8\pi \).
   \[
   108 \text{ square units} \times 8\pi \text{ units} = 864\pi \text{ cubic units}
   \]

4. Use the formula for the volume of a cone, \( \frac{1}{3} \pi r^2 h \), to calculate the volume of the cone. Show your work.
   \[
   \frac{1}{3} \pi (12)^2 (18) = \frac{1}{3} \pi (2592)
   \]
   = 864\pi cubic units

5. Compare the volume that you calculated by rotating the right triangle to the volume that you calculated using the formula for the volume of a cone. What do you notice?
   The volume calculations result in the same volume.
Grouping
Have students complete Questions 6 and 7 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 6 and 7
- The centroid is the point of concurrency for the three medians of a triangle. But, the solution only shows two of the medians. Is there a mistake? Explain.
- If two different medians of the triangle were used, would it result in the same volume formula?
- Does it matter which two medians are used for deriving the formula of a cone?

6. Derive the formula for the volume of any cone with radius, \( r \), and height, \( h \), by rotating a right triangle with vertices at \((0, 0)\), \((0, h)\), and \((r, 0)\).

First, I determine the locations of the midpoints of side \( AB \) and side \( AC \).
- Midpoint of side \( AC \):
  \[
  \left( \frac{0 + r}{2}, \frac{0 + 0}{2} \right) = \left( \frac{r}{2}, 0 \right)
  \]
- Midpoint of side \( AB \):
  \[
  \left( \frac{0 + 0}{2}, \frac{0 + h}{2} \right) = \left( 0, \frac{h}{2} \right)
  \]

Next, I determine an equation representing each median.
- The slope of median \( \overline{CD} \) is \(-\frac{h}{2r}\), and the \( y \)-intercept is \(\frac{h}{2r}\).

Slope of median \( \overline{CD} \) = \frac{0 - \frac{h}{2}}{r - 0} = -\frac{1}{\frac{r}{h}} = -\frac{h}{2r}

So, the equation of median \( \overline{CD} \) is \( y = \frac{-h}{2r}(x) + \frac{h}{2r}\).
- The slope of median \( \overline{BE} \) is \(-\frac{2h}{r}\), and the \( y \)-intercept is \( h \).

Slope of median \( \overline{BE} \) = \frac{0 - h}{1 - 0} = -\frac{h}{1} = -\frac{2h}{r}

So, the equation of median \( \overline{BE} \) is \( y = \frac{-2h}{r}(x) + h\).

Then, I solve the system of equations to determine the coordinates of the centroid.

\[
\begin{align*}
\frac{b}{2r}(x) + \frac{h}{2} &= \frac{-2h}{r} + h \\
\frac{2b}{r} - \frac{h}{2}(x) &= \frac{-2h}{r} + h
\end{align*}
\]

\[
\begin{align*}
\frac{3h}{2r}(x) &= \frac{1}{2}h \\
\frac{x}{6} &= \frac{2h}{3} - \frac{r}{3}
\end{align*}
\]

The centroid of any triangle is located at \( \left( \frac{1}{3r}, \frac{1}{3h} \right) \).

The centroid of a triangle is located at \( \left( \frac{1}{3r}, \frac{1}{3h} \right) \), so the average distance all the points of a triangle are rotated to create a cone is \( 2\pi \left( \frac{1}{3} \right) \), or \( \frac{2}{3}\pi r \).

The area of the triangle is \( \frac{1}{2}r\).

So, the volume of any cone is \( \frac{2}{3}\pi r \times \frac{1}{2}r = \frac{2}{3}\pi r^2h \), or \( \frac{1}{3}\pi r^2h \).
7. Suppose that the entire volume of Lake Erie, which is 116 cubic miles, were contained in a giant cone with a diameter of 16 miles. Determine the height of the cone.

a. Write the formula for the volume of a cone and substitute known values.

Volume of cone: \( \frac{1}{3} \pi (8^2)(h) = 116 \)

b. Solve for the unknown value, the height of the cone. Round your answer to the nearest hundredth.

The height of the cone would be approximately 1.73 miles.

\[
\frac{1}{3} \pi (8^2)(h) = 116 \\
\pi (8^2)(h) = 348 \\
64h \approx 110.77 \\
h \approx 1.73
\]
Problem 3
Students use what they have learned about building formulas for cylinders and cones to construct a formula for the volume of a pyramid. Students then solve application problems involving volume formulas.

Grouping
Have students complete Questions 1 through 8 with a partner. Then have students share their responses as a class.

Guiding Questions for Share Phase, Questions 1 through 8
- Can you think of a cone as a “triangular” cylinder?
- Can you think of a pyramid as a “triangular” prism?
- Where does the fraction $\frac{1}{3}$ come from in the formula for the volume of a cone?

PROBLEM 3 And Now, Pyramids
You can use what you know about the similarities and differences between cylinders and cones to make conjectures about the volume of pyramids.

1. Compare different ways to create cylinders and cones. What similarities and differences are there between creating cylinders and cones:
   a. by stacking?
      Both solid figures are created by stacking circles. A cylinder is created by stacking congruent circles. A cone is created by stacking similar circles that are not congruent.
   b. by rotating?
      For both cylinders and cones, I determine an average point of the polygon that is rotated to create the solid figure. I use this average point to calculate the average distance that all of the points of the polygon are rotated. I, then, multiply the area of the polygon by this average distance to determine the volume formula.
      To create a cylinder, I rotate a rectangle with an average point located at $\left(\frac{1}{2}r, \frac{1}{2}h\right)$.
      To create a cone, I rotate a triangle with an average point located at $\left(\frac{1}{3}r, \frac{1}{3}h\right)$.

2. Analyze the formulas for the volumes of the cylinder and cone.
   \[
   \text{Volume of cylinder} = \pi r^2 h \\
   \text{Volume of cone} = \frac{1}{3}\pi r^2 h
   \]
   a. Which part of each formula describes the area of the base of each solid figure, $B$?
      Explain why.
      The expression $\pi r^2$ describes the area of the base of each solid figure, because the base of each solid is a circle. The area of a circle is given as $A = \pi r^2$.
   b. Which part of each formula describes the height of each solid figure?
      For each solid figure, the variable $h$ in the volume formula represents the height.
   c. Rewrite the volume formulas for each solid figure using the variables $B$ and $h$.
      Cylinder: $V = Bh$
      Cone: $V = \frac{1}{3}Bh$
3. How are the formulas for the volumes of a cylinder and cone similar and different?

Both formulas involve multiplying the area of the base by the height. The volume formula for a cone is \( \frac{1}{3} \) the volume of a cylinder with the same base area and height, so I have to multiply \( B \times h \) by \( \frac{1}{3} \) to obtain the volume of a cone.

4. Now compare different ways to create prisms and pyramids. What similarities and differences are there between creating prisms and pyramids:

a. by stacking?

Both solid figures are created by stacking polygons.
A prism is created by stacking congruent polygons.
A pyramid is created by stacking similar polygons that are not congruent.

b. by rotating?

Prisms and cylinder cannot be created using rotations.

5. Analyze the formula for the volume of a prism.

Volume of prism: \( V = Bh \)

a. Which part of the formula represents the area of the base?

The variable \( B \) represents the area of the base.

b. Which part of the formula represents the height?

The variable \( h \) represents the height.

6. Based on your answers to Questions 1 through 5, what conjecture can you make about the formula for the volume of any pyramid? Explain your reasoning.

I think the volume formula for a pyramid is \( \frac{1}{3} \) the volume of a prism with the same base area and height. A prism is created by stacking congruent polygons, which is similar to creating a cylinder by stacking congruent circles. A pyramid is created by stacking similar polygons that are not congruent, which is similar to creating a cone by stacking similar circles.
7. The Rainforest Pyramid in Galveston, Texas, is a building that is 100 feet high and has a square base with sides that are 200 feet in length. What is the volume of this building?
   a. Determine the area of the base of the pyramid, $B$.
      The area of the square base is 40,000 square feet.
      
      \[ B = 200(200) = 40,000 \]
   
   b. Apply the formula to determine the volume of the pyramid.
      The volume of the pyramid is approximately 1,333,333 cubic feet.
      
      \[ V = \frac{1}{3}Bh = \frac{1}{3}(40,000)(100) = 1,333,333.3 \]

8. The pyramid arena in Memphis, Tennessee, is 321 feet tall and has a square base that is 300 feet on each side. What is the volume of this arena?
   The volume of the Pyramid Arena is 9,630,000 cubic feet.
   
   \[ V = \frac{1}{3}Bh = \frac{1}{3}(300^2)(321) = 9,630,000 \]

Be prepared to share your solutions and methods.
Check for Students’ Understanding

Jody was asked to approximate the volume of a pyramid. She first had to decide how to determine the height of the pyramid. She needs your help.

1. Draw a line segment that represents the height of the pentagonal pyramid.

2. Describe a triangle that can be formed to calculate the height of the pyramid.
   A triangle that can be formed to calculate the height of the pyramid connects the vertex to the center point of the base to a corner of the base.

3. Draw a line segment that represents the height of the square pyramid.

4. Describe a triangle that can be formed to calculate the height of the pyramid.
   A triangle that can be formed to calculate the height of the pyramid connects the vertex to the center point of the base to a midpoint of a side of the base.
5. Jody wanted to determine the height of this polyhedron. She knew that it was the vertical distance from the center point of the base (point \( P \)) to the vertex (point \( T \)), but she wasn’t sure if the triangle she wanted to form should connect to point \( S \), a corner of the base, as shown in figure 1 or point \( A \), the midpoint of the side of the base, as shown in figure 2. She plans to solve for the distance from point \( P \) to point \( T \) using the Pythagorean Theorem.

What advice would you give Jody?
I would tell Jody that either choice would result in the same answer. She should think about what other distances are known in the situation, because she will need to know the length of two sides to use the Pythagorean Theorem. If she is given the length of line segments \( PS \) and \( TS \), she should use figure 1. If she is given the length of line segments \( PA \) and \( TA \), she should use figure 2.
Spheres à la Archimedes

Volume of a Sphere

**LEARNING GOAL**

In this lesson, you will:
- Derive the formula for the volume of a sphere.

**KEY TERMS**

- sphere
- radius of a sphere
- diameter of a sphere
- great circle of a sphere
- hemisphere
- annulus

**ESSENTIAL IDEAS**

- A sphere is the set of all points in three dimensions that are equidistant from a given point called the center.
- The radius of a sphere is a line segment drawn from the center of the sphere to a point on the sphere.
- The diameter of a sphere is a line segment drawn between two points on the sphere passing through the center.
- A cross section of a solid is the two dimensional figure formed by the intersection of a plane and a solid when a plane passes through the solid.
- A great circle of a sphere is a cross section of a sphere when a plane passes through the center of the sphere.
- A hemisphere is half of a sphere bounded by a great circle.
- The volume formula for a sphere is:
  \[ V = \frac{4}{3} \pi r^3, \] where \( V \) is the volume, and \( r \) is the radius of the sphere.

**TEXAS ESSENTIAL KNOWLEDGE AND SKILLS FOR MATHEMATICS**

(11) Two-dimensional and three-dimensional figures. The student uses the process skills in the application of formulas to determine measures of two- and three-dimensional figures. The student is expected to:

- (C) apply the formulas for the total and lateral surface area of three-dimensional figures, including prisms, pyramids, cones, cylinders, spheres, and composite figures, to solve problems using appropriate units of measure.
- (D) apply the formulas for the volume of three-dimensional figures, including prisms, pyramids, cones, cylinders, spheres, and composite figures, to solve problems using appropriate units of measure.
Overview
In this lesson, students use rotation and stacking strategies to derive the formula for the volume of a sphere using a cylinder, cone, and hemisphere with equal heights and radii.
Warm Up

1. What is larger, the volume of Earth or the surface area of Earth?
   The unit of measure for volume is cubic and the unit of measure for area is quadratic. They cannot be compared.

2. Is it possible to compare the amount of sand it would take to fill a model of the Earth and the amount of sand it would take to cover the entire surface of the model? What information would you need? Explain.
   Yes, they could be compared because both tasks involve calculating the volume of sand. You would need to know the depth of sand to cover the surface and you would need to know the radius of Earth.